

**NASA Contractor Report 178370**

**ICASE REPORT NO. 87-60**

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EDDIES IN TWO DIMENSIONAL TURBULENT FLOWS**

**(NASA-CR-178370) ON THE INTERACTION OF  
SMALL AND LARGE EDDIES IN TWO DIMENSIONAL  
TURBULENT FLOWS Final Report (NASA) 45 p  
Avail: NTIS HC A03/MF A01 CSCL 01A**

**N87-29424**

**G3/02 0100129**

**Unclas**

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**Contract No. NAS1-18107  
September 1987**

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NASA Langley Research Center, Hampton, Virginia 23665**

**Operated by the Universities Space Research Association**



**National Aeronautics and  
Space Administration**

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**ON THE INTERACTION OF SMALL AND LARGE  
EDDIES IN TWO DIMENSIONAL TURBULENT FLOWS**

C. Foias<sup>1,4</sup>, O. Manley<sup>2</sup>, and R. Temam<sup>3,4</sup>

**ABSTRACT**

Our aim in this article is to present some results concerning the interaction of small and large eddies in two dimensional turbulent flows. We show that the amplitude of small structures decays exponentially to a small value and we infer from this a simplified interaction law of small and large eddies. Beside their intrinsic interest for the understanding of the physics of turbulence, these results lead to new numerical schemes which will be studied in a separate work.

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This research was supported under the National Aeronautics and Space Administration under NASA Contract No. NAS1-18107 while the third author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.

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## INTRODUCTION

The conventional theory of turbulence in space dimension three implies the existence of a length  $l_d$  which is small in comparison with the macroscopical length  $l_0$  connected to the geometry, and which is such that the eddies of size less than  $l_d$  are damped by the effect of viscosity and become rapidly small in amplitude; the length  $l_d$  is called the Kolmogorov dissipation length [8]. In space dimension two the situation is similar, but  $l_d$  is replaced by the larger length  $l_\chi$  introduced by Kraichnan [9]. It is one of our aims in this article to derive directly from the Navier-Stokes equations and without any phenomenological consideration a mathematically rigorous proof of this property: the exponential decay of the small eddies toward a small value. Note however that the cut-off size between small and large eddies is much smaller than  $l_\chi$  or even  $l_d$ , and this is due in part to the high level of generality allowed here where singular flows can be considered such as those generated by flows in nonsmooth cavities, like the flow in a rectangular cavity. A physical discussion on the necessary cut-off length is presented hereafter.

Our approach is the following one: the Navier-Stokes equations of two dimensional viscous incompressible flows are written as

$$(0.1) \quad \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u - \nabla \bar{w} = f \quad \text{in } \Omega \times \mathbb{R}_+$$

$$(0.2) \quad \nabla \cdot u = 0 \quad \text{in } \Omega \times \mathbb{R}_+$$

where  $u = u(x, t) = \{u_1, u_2\}$  is the velocity vector,  $\bar{w} = \bar{w}(x, t)$  is the pressure,  $f$  represents volume forces,  $\nu > 0$  is the kinematic viscosity. As

usual (0.1)(0.2) are supplemented by boundary conditions which could be for instance

$$(0.3a) \quad u = 0 \quad \text{on} \quad \partial\Omega$$

or

$$(0.3b) \quad u \cdot \underline{v} = 0, \quad \underline{v} \times \text{curl} u = 0 \quad \text{on} \quad \partial\Omega, \quad \underline{v} \quad \text{the unit outward normal on} \quad \partial\Omega,$$

or

$$(0.3c) \quad \Omega = (0, L_1) \times (0, L_2) \quad \text{and} \quad u, \bar{w} \quad \text{are periodic of period} \quad L_1 \quad \text{in the direction} \quad x_1, \quad i = 1, 2.$$

Our emphasis here will be on the space periodic case (0.3c), but the other boundary conditions will be considered as well. In all cases (0.1) - (0.3) reduces to an abstract evolution equation for  $u$  in an appropriate Hilbert space  $H$ :

$$(0.4) \quad \frac{du}{dt} + \nu Au + B(u) = f.$$

The operator  $A$  linear, self-adjoint unbounded positive in  $H$  with domain  $D(A) \subset H$ , is the Stokes operator. Since  $A^{-1}$  is compact self adjoint,  $A$  possesses a complete family of eigenvectors  $w_j$  which is orthonormal in  $H$

$$(0.5) \quad Aw_j = \lambda_j w_j, \quad j = 1, 2, \dots$$

$$0 < \lambda_1 \leq \lambda_2, \dots, \quad \lambda_j \rightarrow \infty \text{ as } j \rightarrow \infty.$$

Of course in the space periodic case (0.3c) the  $w_j$  are directly related to the appropriate sin and cos functions of the Fourier series expansion (see [12]). The operator  $B$  is a quadratic operator;  $B(u) = B(u, u)$ , where  $B(\cdot, \cdot)$  is a bilinear compact operator from  $D(A)$  into  $H$ .

For fixed  $m$  we denote by  $P = P_m$  the projector in  $H$  onto the space spanned by  $w_1, \dots, w_m$ , and we write  $Q = Q_m = I - P_m$ . We set

$$u = p + q, \quad p = Pu, \quad q = Qu,$$

and we show that, after a transient period, and for various norms,  $p$  is comparable to  $u$ , and  $q$  is small in comparison with  $p$  and  $u$  (see Sec. 1).

We then project equation (0.4) on  $PH$  and  $QH$ ; this yields a coupled system of equations for  $p$  and  $q$ :

$$(0.6) \quad \frac{dp}{dt} + \nu Ap + PB(p + q) = Pf$$

$$(0.7) \quad \frac{dq}{dt} + \nu Aq + QB(p + q) = Qf.$$

Since  $q$  is small in comparison with  $p$  one can speculate that  $B(q, q) = B(q)$  is small in comparison with  $B(p, q)$  and  $B(q, p)$  and that these quantities are small in comparison with  $B(p, p) = B(p)$ . Also the relaxation time for the linear part of (0.7) of the order of  $(\nu \lambda_{m+1})^{-1}$  is much smaller than that of (0.6) which is of order  $(\nu \lambda_1)^{-1}$ . This suggests that an acceptable approximation to (0.7) is given by

$$(0.8) \quad \nu Aq + QB(p) = Qf.$$

This leads us to introduce in  $H$  the finite dimensional manifold  $M_0$  with equation

$$(0.9) \quad \begin{cases} q = \Phi_0(p) = (\nu A)^{-1}(Qf - QB(p)) \\ p = Pu, \quad q = Qu. \end{cases}$$

It is one of our aims to justify this approximation: for large times, i.e., after a sufficiently long transient period, the ratio of  $q$  to  $u$  is of the order of  $\frac{\lambda_1}{\lambda_{m+1}}$  for large  $m$ , whereas the distance of  $q$  to  $M_0$ , (compared to a quantity of the order of  $u$ ), is of the order of  $\left(\frac{\lambda_1}{\lambda_{m+1}}\right)^{3/2}$  for large  $m$ . The proof of this result appears in Sec. 2. Hence, for large time, an orbit  $u(t) = p(t) + q(t)$  corresponding to any solution of (0.4) becomes closer to  $M_0$  than to the linear space  $q = 0$ . We intend in a subsequent work to construct a whole family of explicitly defined manifolds  $M_j$  providing better and better approximations to the orbits as  $j$  increases<sup>(1)</sup>. The manifold  $M_0$  (as well as the future manifolds  $M_j$ ) plays the role of approximate inertial manifolds to the two dimensional Navier-Stokes equations and constitutes a substitute to them in situations where we cannot prove the existence of such manifolds.

In Sec. 3 we recall and improve significantly a result in [7]: this leads us to introduce a Lipschitz manifold  $\Sigma$  of finite dimension like  $M_0$ ,

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<sup>(1)</sup>C. Foias, O. Manley, and R. Temam, Article in preparation.

to which all the orbits of (0.4) remain eventually at a distance less than  $\exp(-c \lambda_1 / \lambda_{m+1})$ . Hence  $\sum$  provides a much better approximation than  $M_0$ . but, on the contrary, the proof of existence is nonconstructive and does not provide an explicit expression like (0.9). It provides nevertheless an interesting complementary aspect. Let us mention also that another type of approximate manifold containing all the stationary solutions has been exhibited by E. Titi<sup>(1)</sup>.

This article ends with an Appendix providing a technical but totally new method of estimating certain norms of the solutions of an evolution equation like (0.4): taking advantage of the analyticity in time of the solutions, we estimate the domain of analyticity in the complex time plane and using Cauchy's formula, we readily deduce estimates on the derivatives  $d^k u / dt^k$  from the estimates on  $u$  in the domain of analyticity; these estimates on the time derivatives of  $u$  are much sharper than those obtained by real variable methods.

The results presented here were announced in [0]. We intend in a subsequent work to derive approximate manifolds of higher order than  $M_0$  and to study the three dimensional case.

## 1. FAST DECAY OF SMALL EDDIES

In Secs. 1.1 and 1.2 we briefly recall the functional setting of the Navier-Stokes equations and some useful estimates. Then in Sec. 1.3 we derive the estimates on the magnitude of the small eddies.

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<sup>(1)</sup>E. Titi, Article in preparation.



### 1.1 Preliminaries

As we recalled in the Introduction, the Navier-Stokes equations (0.1)(0.2) associated to one of the boundary conditions (0.3) is equivalent to an evolution equation

$$(1.1) \quad \frac{du}{dt} + \nu Au + B(u) = f$$

in an appropriate Hilbert space  $H$ . Here  $f \in H$ ,  $\nu > 0$ ,  $A$  is a linear self-adjoint positive operator with domain  $D(A) \subset H$ , and whose inverse  $A^{-1}$  is compact; we have  $B(u) = B(u, u)$  where  $B(\cdot, \cdot)$  is a bilinear compact operator from  $D(A)$  (endowed with the norm  $|A \cdot|$ ) into  $H$ ;  $H$  is a Hilbert subspace of  $L^2(\Omega)^2$ . Its norm and scalar product are denoted  $|\cdot|$ ,  $(\cdot, \cdot)$  as those of  $L^2(\Omega)^2$  or  $L^2(\Omega)$ ; for the details see [11][12].

We recall that for  $u_0$  given in  $H$  the initial value problem (1.1)(1.2):

$$(1.2) \quad u(0) = u_0,$$

possesses a unique solution  $u$  defined for all  $t > 0$  and such that

$$(1.3) \quad u \in C(\mathbb{R}_+; H) \cap L^2(0, T; V), \quad \forall T > 0;$$

here  $V = D(A^{1/2})$  and the norm  $|A^{1/2} \cdot| = \|\cdot\|$  on  $V$  is equivalent to the  $L^2$  norm of  $\text{grad } u$ . If  $u_0 \in V$  then

$$(1.4) \quad u \in C(\mathbb{R}_+; V) \cap L^2(0, T; D(A)), \quad \forall T > 0.$$

In both cases ( $u_0 \in H$  or  $V$ ),  $u(\cdot)$  is analytic in  $t$  with values in  $D(A)$ ; the domain of analyticity of  $u$  in the complex plane  $\mathbb{C}_t$  comprises a band around  $\mathbb{R}_+$  and is described in more details in the Appendix.

It is useful here to reproduce some a priori estimates verified by the solutions  $u$  of (1.1)(1.2). Before that we recall some inequalities (continuity properties) concerning  $B$  (see [7]): for every  $u, v, w \in D(A)$ :

$$(1.5) \quad |B(u, v)| \leq c_1 \begin{cases} |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av| \\ |u|^{1/2} |Au|^{1/2} \|v\| \end{cases}$$

$$(1.6) \quad |(B(u, v), w)| \leq c_2 |u|^{1/2} \|u\|^{1/2} \|v\| |w|^{1/2},$$

where  $c_1, c_2$  like the quantities  $c_1, c_1'$ , which will appear subsequently, are dimensionless constraints<sup>(1)</sup>. Also we recall from [1][3] the inequality

$$(1.7) \quad \|\phi\|_{L^\infty(\Omega)}^2 \leq c_3 \|\phi\| \left(1 + \log \frac{|A\phi|^2}{\lambda \|\phi\|}\right), \quad \forall \phi \in D(A),$$

from which we deduce that

$$|B(u, v)| \leq |(u \cdot \nabla) v| \leq \begin{cases} \|u\|_{L^\infty(\Omega)} \|\nabla v\| \\ \|u\| \|\nabla v\|_{L^\infty(\Omega)} \end{cases}$$

and using (1.7)

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(1) These constants can be absolute constants or they may depend on the shape of  $\Omega$ : by this we mean that they are invariant by translation or homothety of  $\Omega$ .

$$(1.8) \quad |B(u,v)| \leq c_4 \begin{cases} \|u\| \|v\| (1 + \log \frac{|Au|^2}{\lambda_1 \|u\|^2})^{1/2} \\ |u| |Av| (1 + \log \frac{|A^{3/2}v|^2}{\lambda_1 |Av|^2})^{1/2} \end{cases}.$$

## 1.2 Behavior of Small Eddies

As mentioned in the Introduction we fix an integer  $m \in \mathbb{N}$  and denote by  $P = P_m$  the projector in  $H$  onto the space spanned by the first  $m$  eigenvectors of  $A$ ,  $w_1, \dots, w_m$ ; we set also  $Q = Q_m = I - P_m$ , and for the sake of simplicity

$$(1.9) \quad \lambda = \lambda_m, \quad \Lambda = \lambda_{m+1}.$$

We write  $p = Pu$ ,  $q = Qu$ ;  $p$  represents a superposition of "large eddies" of size larger than  $\lambda_m^{-1/2}$ , and  $q$  represents "small eddies" of size smaller than  $\lambda_{m+1}^{-1/2}$ . By projecting (1.1) on  $PH$  and  $QH$  we find since  $PA = AP$  and  $QA = AQ$ :

$$(1.10) \quad \frac{dp}{dt} + \nu Ap + PB(p + q) = Pf$$

$$(1.11) \quad \frac{dq}{dt} + \nu Aq + QB(p + q) = Qf.$$

We take the scalar product of (1.10) with  $q$  in  $H$ :

$$(1.12) \quad \frac{1}{2} \frac{d}{dt} |q|^2 + \nu \|q\|^2 = (Qf, q) - (B(p + q), q).$$

Thanks to the orthogonality property

$$(1.13) \quad (B(\phi, \psi), \psi) = 0, \quad \forall \phi, \psi \in V,$$

the right hand side of (1.11) reduces to

$$(Qf, q) - (B(p, p), q) - (B(q, p), q).$$

Using (1.6) and Schwarz inequality we majorize it by

$$\begin{aligned} & |Qf| |q| + c_4 \|p\|^2 |q| \left(1 + \log \frac{|Ap|^2}{\lambda_1 \|p\|^2}\right)^{1/2} + c_2 |q| \|q\| \|p\| \\ & \leq (\text{since } \|p\| \leq \|u\|) \\ & \leq |Qf| |q| + c_4 \|p\|^2 |q| \left(1 + \log \frac{|Ap|^2}{\lambda_1 \|p\|^2}\right)^{1/2} + c_2 \Lambda^{-1/2} \|q\|^2 \|u\|. \end{aligned}$$

We denote now by  $M_0$  (resp.  $M_1$ ,  $M_2$ ) a bound of  $|u|$  (resp.  $\|u\|$ ,  $|Au|$ ), on the interval of time  $I = (t_0, \infty)$  under consideration

$$(1.14) \quad M_0 = \sup_{s \in I} |u(s)|, \quad M_1 = \sup_{s \in I} \|u(s)\|, \quad M_2 = \sup_{s \in I} |Au(s)|;$$

we observe that

$$|Ap|^2 \leq \lambda_m \|p\|^2 = \lambda \|p\|^2$$

and set

$$(1.15) \quad L = \left(1 + \log \frac{\lambda_{m+1}}{\lambda_1}\right).$$

We obtain

$$(1.16) \quad \frac{d}{dt} |q|^2 + (2\nu - c_2 \Lambda^{-1/2} M_1) \|q\|^2 \leq |Qf| |q| + c_4 M_1^2 L^{1/2} |q|.$$

Hence, assuming that  $c_2 \Lambda^{-1/2} M_1 \leq \nu$ , i.e.,

$$(1.17) \quad \lambda_{m+1} = \Lambda \geq \left( \frac{2c_2 M_1}{\nu} \right)^2$$

(1.16) yields

$$\begin{aligned} \frac{d}{dt} |q|^2 + \frac{3\nu}{2} \|q\|^2 &\leq \Lambda^{-1/2} (|Qf| + (4M_1^2 L^{1/2}) \|q\|) \\ &\leq \frac{\nu}{2} \|q\|^2 + \frac{1}{\nu\Lambda} (|Qf|^2 + (c_4^2 M_1^4 L)) \end{aligned}$$

$$(1.18) \quad \frac{d}{dt} |q|^2 + \nu \|q\|^2 \leq \frac{1}{\nu\Lambda} (|Qf|^2 + c_4^2 M_1^4 L)$$

$$(1.19) \quad \frac{d}{dt} |q|^2 + \nu\Lambda |q|^2 \leq \frac{1}{\nu\Lambda} (|Qf|^2 + c_4^2 M_1^4 L).$$

We infer easily from (1.19) that for  $t \geq t_1$ ,  $t_1, t \in I$ :

$$(1.20) \quad |q(t)|^2 \leq |q(t_1)|^2 \exp(-\nu\Lambda(t - t_1)) + \frac{1}{\nu^2 \Lambda^2} (|Qf|^2 + c_4^2 M_1^4 L).$$

Before interpreting this inequality, we derive a similar inequality for the  $(H^1) \vee$  norm. Taking the scalar product of (1.11) with  $Aq$  in  $H$  we find

$$\frac{1}{2} \frac{d}{dt} \|q\|^2 + \nu |Aq|^2 = (Qf, Aq) - (B(p + q), Aq).$$

We expand and use Schwarz inequality and (1.6) - (1.8) to majorize the right

hand side of this equation by

$$\begin{aligned}
 & |Qf| |Aq| + c_2 \|p\| L^{1/2} |Aq| (\|p\| + \|q\|) \\
 & + c_4 |q|^{1/2} |Aq|^{3/2} (\|p\| + \|q\|) \\
 & \leq \text{(with Young's inequality)} \\
 & \leq \frac{\nu}{2} |Aq|^2 + \frac{1}{\nu} |Qf|^2 + \frac{c_1^4 M_1^4 L}{\nu} + \frac{c_2^2}{\nu^3} M_0^2 M_1^4.
 \end{aligned}$$

Thus,

$$(1.21) \quad \frac{d}{dt} \|q\|^2 + \nu |Aq|^2 \leq c_3 \left( \frac{1}{\nu} |Qf|^2 + \frac{M_1^4 L}{\nu} + \frac{M_0^2 M_1^4}{\nu^3} \right)$$

$$(1.22) \quad \frac{d}{dt} \|q\|^2 + \nu \Lambda \|q\|^2 \leq c_3 \left( \frac{1}{\nu} |Qf|^2 + \frac{M_1^4 L}{\nu} + \frac{M_0^2 M_1^4}{\nu^3} \right)$$

and we conclude that

$$\begin{aligned}
 (1.23) \quad & \|q(t)\|^2 \leq \|q(t_1)\|^2 \exp(-\nu \Lambda (t - t_1)) \\
 & + \frac{c_3}{\nu \Lambda} \left( \frac{1}{\nu} |Qf|^2 + \frac{M_1^4 L}{\nu} + \frac{M_0^2 M_1^4}{\nu^3} \right).
 \end{aligned}$$

In (1.20) and (1.23) we can bound  $|q(t_1)|^2$  and  $\|q(t_1)\|^2$  by  $M_0^2$  and  $M_1^2$  respectively. Then after a time depending only on  $M_0$  (or  $M_1$ ),  $\nu$  and  $\Lambda = \lambda_{m+1}$ , the term involving  $t$  becomes negligible and we obtain

$$(1.24) \quad \begin{cases} |q(t)|^2 \leq \frac{2}{v^2 \Lambda^2} (|Qf|^2 + c_4^2 M_1^4 L), \\ \|q(t)\|^2 \leq \frac{2c_3}{v^2 \Lambda} (|Qf|^2 + M_1^4 L + \frac{M_0^2 M_1^4}{v^2}) \end{cases}$$

for  $t$  large. Alternatively, denoting by  $\kappa, \kappa_1, \kappa_1'$ , some quantities which depend only on the data  $v, f, \Omega$ , and  $M_0, M_1, M_2$ , we rewrite (1.24) as

$$(1.25) \quad |q(t)|^2 \leq \kappa L \delta^2, \quad \|q(t)\|^2 \leq \kappa L \delta \quad \text{for } t \text{ large,}$$

$$\delta = \frac{\lambda_1}{\Lambda} = \frac{\lambda_1}{\lambda_{m+1}}, \quad L = 1 + \log \frac{\lambda_{m+1}}{\lambda_1}.$$

Using also the results in the Appendix we conclude the following

**Theorem 1.1:** We assume that  $m$  is sufficiently large so that (1.17) holds. Then for any orbit of (1.1), after a time  $t_*$  which depends only on the initial value  $u(0) = u_0$ , the small eddies component of  $u, q = Q_m u$ , is small in the following sense

$$(1.26) \quad \begin{aligned} |q(t)| &\leq \kappa_0 L^{1/2} \delta, \quad \|q(t)\| \leq \kappa_1 L^{1/2} \delta^{1/2} \\ |q'(t)| &\leq \kappa_0' L^{1/2} \delta \end{aligned}$$

$$|Aq(t)| \leq \kappa_2 L^{1/2}, \quad t \geq t_*.$$

The first two inequalities in (1.26) follow from (1.25); the third one follows

from (1.25) and the analog of (A.15) for  $q$  <sup>(1)</sup>. The fourth inequality is obtained by writing

$$vAq = Qf - q' - QB(p + q)$$

$$|Aq| \leq \frac{1}{v} |Qf| + \frac{1}{v} |q'| + \frac{1}{v} |QB(p + q)|$$

and utilizing (1.5), (1.6), (1.8).

In Sec. 1.3 hereafter we intend to provide a more explicit form of the constants  $\kappa$  in the case of space periodic flows.

### 1.3 The Space Periodic Case

We first review the well-known a priori estimates on the solutions of (1.1). This will yield more explicit expressions for  $M_0, M_1, M_2$ .

We take the scalar product of (1.1) with  $u$  in  $H$ ; using the orthogonality property (1.13) we obtain

$$\frac{1}{2} \frac{d}{dt} |u|^2 + v \|u\|^2 = (f, u) \leq |f| |u|$$

$$\leq \lambda_1^{-1/2} |f| \|u\|$$

$$\leq \frac{v}{2} \|u\|^2 + \frac{1}{2v\lambda_1} |f|^2$$

$$(1.26) \quad \frac{d}{dt} |u|^2 + v \|u\|^2 \leq \frac{1}{v\lambda_1} |f|^2$$

---

<sup>(1)</sup>Note that  $q$  is analytic in the same region of the complex plan as  $u$ .



**Theorem 2.1:** For  $t$  sufficiently large,  $t \geq t_*$ , any orbit of (1.1) remains at a distance in  $H$  of  $P_m H$  of the order of  $\kappa L^{1/2} \delta$  and at a distance in  $H$  of  $M_0$  of the order of  $K L \delta^{3/2}$ . In the norm of  $V$ , the corresponding distances are of order  $\kappa \delta^{1/2} L^{1/2}$  and  $\kappa L \delta$ ; the constants  $\kappa$  depend on the data  $v, \lambda_1, |f|$ , and  $t_*$  depends on these quantities and on  $R_0$ , when  $|u(0)| \leq R_0$ .

### 3. A NONCONSTRUCTIVE RESULT

Our aim in this last section is to exhibit a manifold  $\Sigma$  which is Lipschitz, has finite dimension, and captures the solutions of (1.1) in a much narrower neighborhood than  $M_0$  does. However, the existence of  $\Sigma$  is proved in a nonconstructive way, by opposition with the very simple and explicit equation (2.2) available for  $M_0$ . Secs. 3.1 and 3.2 provide preliminary results and Sec. 3.3 contains the main one.

#### 3.1 Quotient of Norms

We consider two solutions  $u, v$  of (1.1) and set  $w = u - v$ :

$$(3.1) \quad \frac{du}{dt} + v Au + B(u) = f, \quad u(0) = u_0,$$

$$(3.2) \quad \frac{dv}{dt} + v Av + B(v) = f, \quad v(0) = v_0,$$

$$(3.3) \quad \frac{dw}{dt} + v Aw + B(u, w) + B(w, v) = 0.$$

Let  $\sigma$  denote the quotient of norms  $\|w\|^2 / |w|^2$ ; then

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{2((w', w))}{|w|^2} - \frac{2\|w\|^2}{|w|^4} (w', w) = \frac{2}{|w|^2} (w', Aw - \sigma w) \\ &= - \frac{2}{|w|^2} (vAw + B(u, w) + B(w, v), Aw - \sigma w). \end{aligned}$$

Since  $(Aw, Aw - \sigma w) = |Aw - \sigma w|^2$ , we conclude, using (1.5), that

$$\begin{aligned} \frac{d\sigma}{dt} + \frac{2v}{|w|^2} |Aw - \sigma w|^2 &= \\ &= - \frac{2}{|w|^2} (B(u, w) + B(w, v), Aw - \sigma w) \\ &\leq \frac{2}{|w|^2} |Aw - \sigma w| (|B(u, w)| + |B(w, v)|) \\ &\leq \frac{2c_1}{|w|^2} |Aw - \sigma w|^2 (|u|^{1/2} |Au|^{1/2} \|w\| + |w|^{1/2} |Aw|^{1/2} \|v\|) \\ &\leq \frac{v}{|w|^2} |Aw - \sigma w|^2 + \frac{2c_1^2}{v} (|u| |Au| + \|v\| |Av| \lambda_1^{-1/2}) \sigma. \end{aligned}$$

Hence

$$(3.4) \quad \frac{d\sigma}{dt} + \frac{v}{|w|^2} |Aw - \sigma w|^2 \leq \rho \sigma$$

where

$$(3.5) \quad \rho = \rho_u + \rho_v, \quad \rho_u = \frac{2c_1^2}{v\lambda_1^{1/2}} \|u\| |Au|.$$

By integration of the differential inequality  $\sigma' \leq \rho \sigma$ , we find that for

$$t_1 < t < \tau < t_1 + T$$

$$(3.6) \quad \frac{\|w(\tau)\|^2}{|w(\tau)|^2} \leq \frac{\|w(t)\|^2}{|w(t)|^2} \exp\left(\int_t^\tau \rho(s) ds\right).$$

Now we estimate the integral of  $\rho$  in terms of the data; as in (1.16) we assume that on the interval of time under consideration

$$(3.7) \quad \|u(t)\| \leq M_1, \quad \|v(t)\| \leq M_1.$$

With an appropriate value of  $M_1$  (3.7) will be valid on some finite interval of time  $[0, T]$ , or on some interval of time  $(t_0, \infty)$ , once the orbits have entered the absorbing set.

We have

$$\begin{aligned} \int_t^\tau \rho_u ds &\leq \frac{2c_1}{v\lambda_1^{1/2}} \int_t^\tau \|u\| |Au| ds \\ &\leq \frac{2c_1^2}{v\lambda_1^{1/2}} M_1 (\tau - t)^{1/2} \left(\int_t^\tau |Au|^2 ds\right)^{1/2}. \end{aligned}$$

An estimate on  $Au$  is obtained by taking the scalar product of (3.1) with  $Au$  in  $H$ :

$$\frac{d}{dt} \|u\|^2 + 2\nu |Au|^2 = -2(B(u), Au) - 2(f, Au)$$

$$\leq 2|B(u)| |Au| + 2|f| |Au|$$

$$\leq (\text{with (1.5)})$$

$$\leq 2c_1 |u|^{1/2} \|u\| |Au|^{1/2} + 2|f| |Au|$$

$$\leq \nu |Au|^2 + \frac{c_1'}{v^3} |u|^2 \|u\|^4 + \frac{2}{v} |f|^2$$

$$(3.8) \quad \frac{d}{dt} \|u\|^2 + \nu |Au|^2 \leq \frac{1}{v} |f|^2 + \frac{c_1'}{v^3 \lambda_1} M_1^6.$$

Thus,

$$(3.9) \quad \begin{aligned} \int_{t_1}^{t_1+\tau} |Au|^2 ds &\leq \frac{\|u(t_1)\|^2}{v} + \frac{\tau}{v^2} (|f|^2 + \frac{c_1'}{v^2 \lambda_1} M_1^6) \\ \int_{t_1}^{t_1+\tau} |Au|^2 ds &\leq \frac{M_1^2}{v} + \frac{\tau |f|^2}{v^2} + \frac{c_1' \tau}{v^4 \lambda_1} M_1^6 \end{aligned}$$

and

$$(3.10) \quad \begin{aligned} \int_t^\tau \rho_u ds &\leq \frac{1}{2} (\tau - t)^{1/2} \kappa_3 \\ \kappa_3 &= \frac{c_2'}{v \lambda_1^{1/2}} M_1 \left( \frac{M_1^2}{v} + \frac{\tau |f|^2}{v^2} + \frac{\tau M_1^6}{v^4 \lambda_1} \right)^{1/2}. \end{aligned}$$

Since the estimates on  $v$  and  $\rho_v$  are the same, we have

$$(3.11) \quad \int_t^\tau \rho ds \leq (\tau - t)^{1/2} \kappa_3.$$

### 3.2 The squeezing property

The squeezing property is an important property of the solutions of the Navier-Stokes equations which has been introduced in [6]. A stronger form of it, called the strong squeezing property or the cone property was proven in [4] for some other, more strongly dissipative equations. For the two dimensional Navier-Stokes equations, we derive here a form of the squeezing property sharper than in [6].

We take the scalar product of (3.3) with  $w$  in  $H$  and thanks to (1.13), (1.16) we find

$$\begin{aligned}
 \frac{d}{dt} |w|^2 + 2\nu \|w\|^2 &= -2b(w, v, w) \\
 &\leq 2c_2 |w| \|w\| \|v\| \\
 &\leq \nu \|w\|^2 + \frac{c_2^2}{\nu} |w|^2 \|v\|^2 \\
 &\leq \nu \|w\|^2 + \frac{c_2^2}{\nu} M_1^2 |w|^2 \\
 (3.12) \quad \frac{d}{dt} |w|^2 + \left( \nu \frac{\|w\|^2}{|w|^2} - \frac{c_2^2}{\nu} M_1^2 \right) |w|^2 &\leq 0.
 \end{aligned}$$

We consider  $t_0, t, 0 < t < t_0 \leq T$  and write, using (3.6)(3.11)

$$(3.13) \quad \gamma_0 = \frac{\|w(t_0)\|^2}{|w(t_0)|^2} \leq \exp(\kappa_3(t_0 - t)^{1/2}) \frac{\|w(t)\|^2}{|w(t)|^2}.$$

Thus,

$$(3.14) \quad \frac{d}{dt} |w|^2 + (\nu \gamma_0 \exp(-\kappa_3 t_0^{1/2}) - \frac{c_2^2}{\nu} M_1^2) |w|^2 \leq 0$$

and by integration

$$(3.15) \quad |w(t_0)|^2 \leq |w(0)|^2 \exp(-\nu \gamma_0 t_0 \exp(-\kappa_3 t_0^{1/2}) + \frac{c_2^2}{\nu} M_1^2 t_0).$$

Now if  $|Q_m w(t_0)| > |P_m w(t_0)|$ , we write

$$\gamma_0 = \frac{\|P_m w(t_0)\|^2 + \|Q_m w(t_0)\|^2}{|P_m w(t_0)|^2 + |Q_m w(t_0)|^2} \\ \geq \frac{\|Q_m w(t_0)\|^2}{2|Q_m w(t_0)|^2} \geq \frac{\lambda_{m+1}}{2}$$

and

$$|w(t_0)|^2 \leq |w(0)|^2 \exp(-\nu \lambda_{m+1} \kappa_5 t_0 + \kappa_4 t_0) \quad (3.16) \\ \kappa_4 = \frac{c_2^2}{\nu} M_1^2, \quad \kappa_5 = \frac{1}{2} \exp(-\kappa_3 t_0^{1/2}).$$

Of course the interval  $(0, t_0)$  can be replaced by any interval  $(t_1, t_1 + t_0)$  on which the bound (3.7) is valid.

In conclusion (this is the squeezing property), whenever (3.7) is valid on some interval  $(t_1, t_1 + t_0)$ , then  $w = u - v$  satisfies one of the following conditions:

$$(3.17a) \quad |Q_m w(t_0 + t_1)| \leq |P_m w(t_0 + t_1)|$$

or

$$(3.17b) \quad |w(t_0 + t_1)|^2 \leq |w(t_1)|^2 \exp(-\nu \lambda_{m+1} \kappa_5 t_0 + \kappa_4 t_0).$$

Since  $\kappa_4, \kappa_5$  are independent of  $m$ , the exponential term in (3.17b) can be made arbitrarily small by choosing  $m$  sufficiently large; we will take advantage of this remark in Sec. 3.3.

Of slightly more explicit form of  $\kappa_4, \kappa_5$  can be derived by using the Grashof number  $G = |f|/\nu^2 \lambda_1$  and the Reynolds type number  $R_n = M_1/\nu \lambda_1^{1/2}$ . We find  $(\tau = t_0)$ :

$$\begin{aligned}
 \kappa_3 &= c_2^2 R_n (\nu \lambda_1)^{1/2} (R_n^2 + t_0 \nu \lambda_1 G^2 + t_0 \nu \lambda_1 R_n^6)^{1/2} \\
 (3.18) \quad \kappa_4 &= c_2^2 R_n^2 (\nu \lambda_1) \\
 \kappa_5 &= \frac{1}{2} \exp((-c_2^2 R_n (\nu \lambda_1 t_0)^{1/2} (R_n^2 + t_0 \nu \lambda_1 G^2 + t_0 \nu \lambda_1 R_n^6)^{1/2}).
 \end{aligned}$$

In the space periodic case we have seen that, for large times, we can take  $M_1 = (2|f|G)^{1/2}$ . Then  $R_n = \sqrt{2} G$  and the above quantities become

$$\begin{aligned}
 \kappa_3 &= c_3^2 (\nu \lambda_1)^{1/2} (G^4 + t_0 \nu \lambda_1 G^4 + t_0 \nu \lambda_1 G^8)^{1/2} \\
 (3.19) \quad \kappa_4 &= 2c_2^2 (\nu \lambda_1) G^2 \\
 \kappa_5 &= \frac{1}{2} \exp(-c_3^2 (\nu \lambda_1 t_0)^{1/2} (G^4 + t_0 \nu \lambda_1 G^4 + t_0 \nu \lambda_1 G^8)^{1/2}).
 \end{aligned}$$

### 3.3 The Approximate Manifold

We denote by  $S(t)$ ,  $t > 0$  the operator in  $H$ :  $u_0 \rightarrow u(t)$ , where  $u(\cdot)$  is the unique solution of (1.1) satisfying  $u(0) = u_0$ . The operators  $S(t)$ ,  $t \geq 0$ , form a semigroup in  $H$ .

The squeezing property tells us that if  $u(\cdot), v(\cdot)$  are two solutions of (1.1) lying in the ball  $\{\phi \in v, \|\phi\| \leq M_1\}$ , for  $0 \leq t \leq T$ , then at each time  $t \in [0, T]$  and for every  $m \in \mathbb{N}$ , we have either

$$|Q_m(S(t)u_0 - S(t)v_0)| \leq |P_m(S(t)u_0 - S(t)v_0)|$$

or

$$|S(t)u_0 - S(t)v_0| \leq |u_0 - v_0| \exp \frac{1}{2} (-\nu\lambda_{m+1}\kappa_5 t_0 + \kappa_4 t_0)$$

$\kappa_4, \kappa_5$  as above.

Now we choose  $t_0 \in [0, T]$ ,  $m \in \mathbb{N}$  and consider a subset  $\sum = \sum (m)$  of

$$S(t_0) \{u_0 \in V, \|u_0\| \leq M_1\}$$

which is maximal under the property

$$(3.20) \quad |Q_m(u - v)| \leq |P_m(u - v)|.$$

By this we mean that if  $u \in \sum (m)$  then

$$\{v \in V, v \text{ satisfies (3.20)}\} \supseteq \sum (m).$$

The existence of such a maximal set is easy.

We then apply the squeezing property: whenever  $\|u(s)\| \leq M_1$ , we see that  $S(t_0)u(s) = u(t_0 + s)$  either belongs to  $\sum (m)$ , i.e.,

$$|Q_m(S(t_0)u(s) - S(t_0)\phi)| \leq |P_m(S(t_0)u(s) - S(t_0)\phi)|,$$

for some  $\phi \in V$  such that  $\|\phi\| \leq M_1$  and  $S(t_0)\phi \in \sum (m)$  or, if not, then for every such  $\phi$

$$|S(t_0)u(s) - S(t_0)\phi|^2 \leq |u(s) - \phi|^2 \exp(-\nu\lambda_{m+1}\kappa_5 t_0 + \kappa_4 t_0)$$



$$\leq \frac{4M_1^2}{\lambda_1} \exp(-v\lambda_{m+1}\kappa_5 t_0 + \kappa_4 t_0).$$

In all cases the distance of  $S(t_0)u(s)$  to  $\sum (m)$  is bounded by

$$\frac{2M_1}{\lambda_1^{1/2}} \exp\left(\frac{t_0}{2} (\kappa_4 - v\lambda_{m+1}\kappa_5)\right).$$

We can choose  $t_0 = (v\lambda_1)^{-1}$  and the bound becomes

$$\frac{2M_1}{\lambda_1^{1/2}} \exp\left(-\frac{\kappa_5}{4} \frac{\lambda_{m+1}}{\lambda_1}\right)$$

provided that

$$(3.21) \quad \frac{\lambda_{m+1}}{\lambda_1} \geq \frac{2\kappa_4}{\kappa_5 v \lambda_1}.$$

By translation in time  $(t \rightarrow t - t_*)$ , we conclude that once the orbit  $u$  has entered the absorbing set  $\{\|\phi\| \leq M_1\}$ , which happens for  $t \geq t_* = t_*(R_0)$  (for  $|u(0)| \leq R_0$ ), the distance of  $S(t)u_0$  to  $\sum (m)$  is bounded by a given quantity  $E$ ,

$$(3.22) \quad \text{dist}_H(S(t)u_0, \sum (m)) \leq E$$

provided  $t \geq t_* + (v\lambda_1)^{-1}$ , and

$$\exp\left(-\frac{\kappa_5}{4} \frac{\lambda_{m+1}}{\lambda_1}\right) \leq E,$$

i.e.,

$$(3.23) \quad \frac{\lambda_{m+1}}{\lambda_1} \geq -\frac{4}{\kappa_5} \log E.$$

By definition the set  $\sum(m)$  enjoys the property that

$$|Q_m(u - v)| \leq |P_m(u - v)|, \quad \forall u, v \in M(m).$$

Hence,  $\sum(m)$  is the graph of a Lipschitz function

$$\psi: P_m \sum(m) \rightarrow Q_m H$$

$$|\psi(P_m u) - \psi(P_m v)| \leq |P_m u - P_m v|, \quad \forall P_m u, P_m v \in P_m \sum(m).$$

By the Kirszbaum extension Theorem [16]  $\psi$  can be extended as a Lipschitz function (with the same constant) from  $P_m H$  into  $Q_m H$ , that we still denote by  $\psi$ . Now  $\psi$  is defined from  $P_m H$  into  $Q_m H$ , and its graph is a Lipschitz manifold above all of  $P_m H$ .

In conclusion we have proved the following theorem

**Theorem 3.1:** If  $m$  is sufficiently large so that (3.21) is satisfied<sup>(1)</sup> then there exists a Lipschitz manifold  $\sum(m)$  of dimension  $m$ , which enjoys the following property: for an solution  $u(\cdot)$  of (1.1), for  $t$  sufficiently large ( $t \geq t_{**}(R_0, v, f, \Omega)$ , for  $|u_0| \leq R_0$ ), the distance in  $H$  of  $u(t)$  to  $\sum(m)$  is majorized by

$$\frac{2M_1}{\lambda_1^{1/2}} \exp\left(-\frac{\kappa_5}{4} \frac{\lambda_{m+1}}{\lambda_1}\right).$$

---

(1)  $\kappa_4, \kappa_5$  as above with  $t_0 = (v\lambda_1)^{-1}$ , and  $M_1$  the radius of an absorbing set in  $V$  for (1.1).

#### **ACKNOWLEDGEMENT**

This work was supported in part by the Applied Mathematical Science Program of the U.S. Department of Energy, Contract DE-AC02-82ER-12049 and Grant DE-FG02-86ER25020 and by the Research Fund of Indiana University. Parts of the work were done during a visit of the second author (OM) at the Université of Paris-Orsay and a time where the third author (RT) enjoyed the hospitality of the Institute for Computer Applications in Science and Engineering (ICASE).

# APPENDIX

## Estimates in the Complex Time Plane

It was proved in [6] (see also [12]) that the solutions to the Navier-Stokes equations are analytic in time; we want to show how one can then use Cauchy's formula to get a priori estimates on the time derivatives of the solutions. The main point in the proof is to determine the width of the band of analyticity of the solution around the real axis  $\mathbb{R}_+$ ; this will follow as in [6,12] from a priori estimates on the solution in the complex plane.

The complex time is denoted  $\zeta = se^{i\theta}$ ;  $\mathbb{H}$ ,  $\mathbb{V}$ ,  $\mathbb{D}(A)$  are the complexified spaces of  $H$ ,  $V$ ,  $D(A)$ ;  $A$ ,  $B$  are extended as linear and bilinear operators respectively from  $\mathbb{D}(A)$  into  $\mathbb{H}$ :

$$(A.1) \quad A(u_1 + iu_2) = Au_1 + iAu_2,$$

$$(A.2) \quad B(u_1 + iu_2, v_1 + iv_2) = B(u_1, v_1) - B(u_2, v_2)$$

$$+ i[B(u_2, v_1) + B(u_1, v_2)]$$

$\forall u = u_1 + iu_2, v = v_1 + iv_2 \in \mathbb{D}(A)$ . The Navier-Stokes equation (1.1) becomes ( $u = u(\zeta)$ ):

$$(A.3) \quad \frac{du}{d\zeta} + \nu Au + B(u) = f$$

$$(A.4) \quad u(0) = u_0.$$

Assuming that  $u_0 \in V$  (or  $\mathcal{M}$ ),  $u|_{\mathbb{R}_+} \in L^\infty(\mathbb{R}_+; \mathcal{M})$  and as in (1.14), we denote by  $M_0, M_1$ , the supremum of  $|u(t)|$  and  $\|u(t)\|$ ,  $t \in \mathbb{R}_+$ . We take the scalar product in  $\mathcal{H}$  of (A.3) with  $Au$ ; we multiply the resulting equation by  $e^{i\theta}$  and take its real part. This yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{ds} \|u(se^{i\theta})\|^2 + v \cos \theta |Au(se^{i\theta})|^2 = \\ (A.5) \quad & = -\operatorname{Re} e^{i\theta} (B(u), Au) - \operatorname{Re} e^{i\theta} (f, Au) \end{aligned}$$

$$\leq |(B(u), Au)| + |f| |Au|.$$

We expand by bilinearity (using (A.2)) and bound the resulting expressions with the help of (1.8):

$$|(B(u), Au)| \leq c \|u\|^2 \left(1 + \log \frac{|Au|^2}{\lambda_1 \|u\|^2}\right)^{1/2} |Au|.$$

Also

$$|f| |Au| \leq \frac{v \cos \theta}{2} |Au|^2 + \frac{|f|^2}{2v \cos \theta}.$$

Hence (with  $u = u(se^{i\theta})$ ):

$$\begin{aligned} (A.6) \quad & \frac{d}{ds} \|u\|^2 + v \cos \theta |Au|^2 \leq \\ & \leq \frac{|f|^2}{v \cos \theta} + c_5 \|u\|^2 |Au| \left(1 + \log \frac{|Au|^2}{\lambda_1 \|u\|^2}\right)^{1/2}. \end{aligned}$$

We write  $z = \frac{|Au|}{\lambda_1^{1/2} \|u\|} \geq 1$  and consider the function

$$z + \phi(z) = -\frac{\lambda_1 v \cos \theta}{2} z^2 + c_5 \|u\| \lambda_1^{1/2} z (1 + \log z^2)^{1/2}.$$

By elementary computations<sup>(1)</sup>

$$(A.7) \quad \phi(z) \leq \frac{c_5^2 \|u\|^2}{2v \cos \theta} \log \left( \frac{4c_5^2 \|u\|^2}{\lambda_1 v^2 \cos^2 \theta} \right), \text{ for } z \geq 1,$$

and (A.6) yields

$$(A.8) \quad \begin{aligned} \frac{d}{ds} \|u\|^2 + \frac{v \cos \theta}{2} |Au|^2 &\leq \\ &\leq \frac{|f|^2}{v \cos \theta} + \frac{c_5^2}{2v \cos \theta} \|u\|^4 \left( \log \frac{4c_5^2 \|u\|^2}{\lambda_1 v^2 \cos^2 \theta} \right). \end{aligned}$$

Setting  $y(s) = \frac{(1 + 4c_5^2)}{\lambda_1 v^2 \cos^2 \theta} (|f|^2 + \|u(\sec^{10})\|^2)$  we infer from (A.8) that

$$\frac{dy}{ds} \leq c_1' \lambda_1 v \cos \theta y^2 \log y,$$

where  $c_1'$  is an appropriate nondimensional constant. As long as  $y(s) \leq 2y_0 = 2y(0)$ , we have

---

(1) Looking for the maximum of  $-\alpha^2 z^2 + \beta^2 (1 + \log z^2)$ , we find

$$\begin{aligned} \beta^2 (1 + \log z^2) &\leq \alpha^2 z^2 + \beta^2 \log \frac{\beta^2}{\alpha^2} \\ z\beta (1 + \log z^2)^{1/2} &\leq \alpha z^2 + \beta z \left( \log \frac{\beta^2}{\alpha^2} \right)^{1/2} \\ &\leq 2\alpha z^2 + \frac{1}{4} \frac{\beta^2}{\alpha} \left( \log \frac{\beta^2}{\alpha^2} \right). \end{aligned}$$

We then choose  $\alpha = \frac{\lambda_1 v \cos \theta}{4}$ ,  $\beta = c_5 \|u\| \lambda_1^{1/2}$ .

$$y' \leq c_1 \lambda_1 v \cos \theta y^2 \log(2y_0)$$

$$y(s) \leq \frac{y_0}{1 - c_1 \lambda_1 v \cos \theta \log(2y_0) s},$$

and this is indeed  $\leq 2y_0$  as long as  $s \leq T_*$ :

$$T_* = \frac{3}{2c_1 \lambda_1 v \cos \theta y_0 \log(2y_0)}.$$

For  $\|u_0\| \leq M_1$ , we replace  $T_*$  by

$$(A.9) \quad T_*(M_1) = \frac{3}{2c_1 \lambda_1 v \cos \theta \left( \frac{G}{\cos^2 \theta} + \frac{M_1^2}{\lambda_1 v^2 \cos^2 \theta} \right) \log 2 \left( \frac{G}{\cos^2 \theta} + \frac{M_1^2}{\lambda_1 v^2 \cos^2 \theta} \right)}.$$

Thus

$$(A.10) \quad \|u(se^{i\theta})\|^2 \leq 2(|f| + \|u_0\|^2) \leq 2(|f| + M_1^2)$$

for

$$0 \leq s \leq \frac{3 \cos \theta}{2c_1 \lambda_1 v \left( G + \frac{M_1^2}{\lambda_1 v^2} \right) + \log 2 \left( \frac{G}{\cos^2 \theta} + \frac{M_1^2}{\lambda_1 v^2 \cos^2 \theta} \right)}$$

and in particular for

$$(A.11) \quad 0 \leq s \leq \frac{3 \cos \theta}{2c_1 \lambda_1 v \left(G + \frac{M_1^2}{\lambda_1 v^2}\right) + \log 4 \left(G + \frac{M_1^2}{\lambda_1 v^2}\right)}$$

when  $\cos^2 \theta \geq \frac{1}{2}$ .

Following the method developed in [6] we conclude that the solution  $u$  of (A.3) (or (1.a)) is analytic in the region

$$(A.12) \quad \Delta(u_0) = \{se^{i\theta}, s \leq \alpha \cos \theta, \cos \theta \geq \frac{\sqrt{2}}{2}\}$$

$$\alpha = \frac{3}{2c_1 \lambda_1 v \left(G + \frac{M_1^2}{\lambda_1 v^2}\right) + \log 4 \left(G + \frac{M_1^2}{\lambda_1 v^2}\right)}$$

which comprises the regions

$$|\operatorname{Im} \zeta| \leq \operatorname{Re} \zeta, \quad 0 < \operatorname{Re} \zeta \leq \frac{\alpha}{2}$$

and

$$(A.13) \quad |\operatorname{Im} \zeta| \leq \frac{\alpha}{2}, \quad \operatorname{Re} \zeta \geq \frac{\alpha}{2}.$$

At any point  $t \in \mathbb{R}_+$ ,  $t \geq \alpha$ , we can apply Cauchy's formula to the circle  $\Gamma$  centered at  $t$  of radius  $\alpha/4$ :

$$(A.14) \quad \frac{d^k u(t)}{dt^k} = \frac{k!}{2\pi i} \int_{\Gamma} \frac{u(\zeta)}{(t-\zeta)^{k+1}} d\zeta.$$

Thus,

$$(A.15) \quad \sup_{t \geq \alpha} \left| \frac{d^k u(t)}{dt^k} \right| \leq \frac{4^k}{\alpha} k! M_0$$



$$(A.16) \quad \sup_{t \geq \alpha} \left\| \frac{d^k u}{dt^k}(t) \right\| \leq \frac{4^k}{\alpha^k} k! M_1.$$

Explicit values of  $M_0$  and  $M_1$  were derived in (1.36) for the two dimensional space periodic case:  $M_1 = (2|f|G)^{1/2}(t \geq t_2)$ . This yields (assuming  $G \geq 1$ ):

$$(A.17) \quad \alpha = \frac{3}{2c_1 \lambda_1^\nu (G + 2G^2) \log 4(G + 2G^2)}$$

$$\alpha \geq \frac{c_2}{\lambda_1^\nu G^2 \log G}$$

and we deduce from (A.15), (A.16) that for  $t$  sufficiently large<sup>(1)</sup>

$$(A.18) \quad \left| \frac{d^k u(t)}{dt^k} \right| \leq c \frac{|f|^{1/2}}{\lambda_1^{1/2}} (|f| \lambda_1)^{k/2} (G^2 \log G)^k$$

$$\left\| \frac{d^k u(t)}{dt^k} \right\| \leq c |f|^{1/2} (|f| \lambda_1)^{k/2} (G^2 \log G)^k.$$

In particular ( $k = 1$ ):

$$(A.19) \quad \left| \frac{du(t)}{dt} \right| \leq c |f| G^2 \log G$$

$$\left\| \frac{du(t)}{dt} \right\| \leq c |f| \lambda_1^{1/2} G^2 \log G, \quad t \geq T_*.$$

This produces an interesting bound on  $|A_u(t)|$  for  $t$  large:

---

(1) This means as in Theorem 1.1 and elsewhere  $t \geq T_*(R_0, \nu, \lambda_1, |f|)$ , for  $|u_0| \leq R_0$ .

$$vAu = f - B(u) - u'$$

$$|Au| \leq \frac{1}{v} |f| + \frac{c_1}{v} |u|^{1/2} \|u\| |Au|^{1/2} + \frac{1}{v} |u'|$$

$$|Au| \leq \frac{2}{v} |f| + \frac{c_1^2}{v^2} |u| \|u\|^2 + \frac{2}{v} |u'|$$

$$\leq c(|f|_{\lambda_1})^{1/2} (G^{1/2} + G + G^{5/2} \log G)$$

(A.20)

$$|Au(t)| \leq c(|f|_{\lambda_1})^{1/2} G^{5/2} \log G, \quad \text{for } t \geq T_*$$

# REFERENCES

- [1] H. Brézis and T. Gallouet, "Nonlinear Schroedinger evolution equation," Nonlinear Analysis Theory Methods and Applications, Vol. 4, 1980, p. 677.
- [2] C. Foias, O. Manley, and R. Temam, "Sur l'interaction des petits et grands tourbillons dans des écoulements turbulents," C. R. Ac. Sc. Paris, 1987.
- [3] C. Foias, O. Manley, R. Temam, and Y. Treve, "Asymptotic analysis of the Navier-Stokes equations," Physica 6D, 1983, pp. 157-188.
- [4] C. Foias, B. Nicolaenko, G. Sell, and R. Temam, "Variétés inertielles pour l'équation de Kuramoto-Sivashinsky," C. R. Ac. Sc. Paris, 301, Série I, 1985, pp. 285-288 and "Inertial Manifolds for the Kuramoto-Sivashinsky equations and an estimate of their lowest dimension," J. Math. Pure Appl., 1988.
- [5] C. Foias and G. Prodi, "Sur le comportement global des solutions non stationnaires des équations de Navier-Stokes en dimension 2," Rend. Sem. Mat. Padova, Vol. 39, 1967, pp. 1-34.
- [6] C. Foias and R. Temam, "Some analytic and geometric properties of the solutions of the Navier-Stokes equations," J. Math. Pure Appl., Vol. 58, 1979, pp. 339-368.

- [7] C. Foias and R. Temam, "Finite parameter approximative structures of actual flows," in Nonlinear Problems: Present and Future, A. R. Bishop, D. K. Campbell, B. Nicolaenko (eds.), North Holland, Amsterdam, 1982.
- [8] A. N. Kolmogorov, C. R. Ac. Sc. URSS, Vol. 30, 1941, p. 301; Vol. 31, 1941, p. 538; Vol. 32, 1941, p. 16.
- [9] R. H. Kraichnan, "Inertial ranges in two dimensional turbulence," Phys. Fluids, Vol. 10, 1967, pp. 1417-1423.
- [10] G. Métivier, "Valeurs propres d'opérateurs définis sur la restriction de systemes variationnels a des sous-espaces," J. Math. Pure Appl., Vol. 57, 1978, pp. 133-156.
- [11] R. Temam, Navier-Stokes Equations, 3rd Revised Ed., North Holland, Amsterdam, 1984.
- [12] R. Temam, Navier-Stokes Equations and Nonlinear Functional Analysis, NSF/CBMS Regional Conferences Series in Appl. Math., SIAM, Philadelphia, 1983.
- [13] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, 1988.

- [14] J. H. Wells and L. R. Williams, Imbeddings and Extensions in Analysis, Springer-Verlag, Heidelberg, New York, 1975.



## Report Documentation Page

1. Report No. NASA CR-178370 ICASE Report No. 87-60		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle ON THE INTERACTION OF SMALL AND LARGE EDDIES IN TWO DIMENSIONAL TURBULENT FLOWS				5. Report Date September 1987	
				6. Performing Organization Code	
7. Author(s) C. Foias, O. Manley, R. Temam				8. Performing Organization Report No. 87-60	
				10. Work Unit No. 505-90-21-01	
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225				11. Contract or Grant No. NAS1-18107	
				13. Type of Report and Period Covered Contractor Report	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665-5225				14. Sponsoring Agency Code	
15. Supplementary Notes Langley Technical Monitor: Richard W. Barnwell  Final Report					
16. Abstract <p>Our aim in this article is to present some results concerning the interaction of small and large eddies in two dimensional turbulent flows. We show that the amplitude of small structures decays exponentially to a small value and we infer from this a simplified interaction law of small and large eddies. Beside their intrinsic interest for the understanding of the physics of turbulence, these results lead to new numerical schemes which will be studied in a separate work.</p>					
17. Key Words (Suggested by Author(s)) fluid mechanics, turbulence, eddies simulation			18. Distribution Statement 02 - Aerodynamics 34 - Fluid Mechanics and Heat Transfer  Unclassified - unlimited		
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified		21. No. of pages 44	22. Price A03	